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A LINEAR MODEL AND ITS PATH OF BALANCED GROWTH

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1. INTRODUÇÃO

In this paper we formalize a linear model of growth where the distinction between investment goods and consumption goods is not established a priori but relies on the type of the demand of the economic agents. We deal with a n-sectoral input-output structure and we show the existence of balanced growth in the context of a dynamic Leontief-type of model closely related to that of DOSSO.

2. THE GENERALIZED MODEL

Let $\in$ be an economy with n sectors: $E_1, \ldots, E_i, \ldots, E_n$. The set $L_n = \{1, \ldots, n\}$ denotes the set of natural subindexes from 1 to n. Let:

$$\rho : i_p \rightarrow i_q, \; p, q \in L_n$$

be the association rule: "E_{i_p} produces capital goods to E_{i_q}"; and let

$$\sigma : i_r \rightarrow i_s, \; r, s \in L_n$$

be the association rule: "E_{i_r} produces consumption goods to E_{i_s}".

Let $\Omega(\in) = \{E_1, \ldots, E_i, \ldots, E_n\}$ be the of all sectors of the economy. Let $I = \{i_1, \ldots, i_\alpha, \ldots, i_k\} \subset L_n$ be a subset of L_n and let

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$J = \{ i_{k+1}, ..., i_\beta, ..., E_n \} \subset L_n$ be the complement of I in L_n, with $k \leqslant n$. Let

$$\Omega (\in, I) = \{ E_{i_1}, ..., E_{i_\alpha}, ... E_{i_k} \}$$

$$\Omega (\in, J) = \{ E_{i_{k+1}}, ..., E_{i_\beta}, ..., E_{i_n} \}$$

be, respectively, the set of sectors of $\in$ which corresponds to the subset I and the set of sectors of $\in$ which corresponds to J. Given a sector $E_i \in \Omega (\in)$ and the association rules, not necessarily injective and/or surjective, the images of E_i under ρ and σ are, respectively:

$$\text{Im}\, \rho (E_i) = \{ E_{i_1}, ..., E_{i_\alpha}, ..., E_{i_k} \}$$

$$\text{Im}\, \sigma (E_i) = E_{i_{\alpha_1}}, ..., E_{i_{\alpha_\theta}}, E_{i_{\beta_1}}, ..., E_{i_{\beta_\lambda}}$$

$$1 \leqq \theta \leqq k$$

$$k + 1 \leqq \lambda \leqq n$$

Then we have the following inequalities:

$$\# \text{Im}\, \rho (E_i) + \# \text{Im}\, (E_i) \leqslant \# \Omega (\in) + \# \Omega (\in) = 2n$$

$$\sum_{i=1}^{n} \{ \# \text{Im}\, (E_i) + \# \text{Im}\, (E_i) \} \leqslant \# \Pi (\in) + \Pi (\in) = 2 n^2$$

where # denotes the cardinality of a countable set and $\Pi (\in)$ is the cartesian product $\Omega (\in) \times \Omega (\in)$.

The sectoral product A_i of the sector E_i is a linear function of the sectoral stock K_i of the sector E_i:

$$A_i = b_i\, k_i, \forall i \in L_n$$

where b_i is the sectoral coefficient output/capital of the sector E_i.

Let h_{ij} be the relative contribution of the production A_j of the sector E_j to the production A_i of the sector E_i. Furthermore, we define:

$$\gamma_j = \sum_{i=1}^{n} h_{ij}, \forall j \in L_n$$

$$\Gamma = \{\gamma_1, ..., \gamma_j, ..., \gamma_n\}$$

We can see that:

$$\theta \leqq \min_{j \in L_n} \Gamma \leqq \max_{j \in L_n} \Gamma \leqq 1$$

Let us suppose that the total output Y as well as the sectoral production A_i of the sector E_i and its sectoral capital stock K_i are continuous functions in relation to the time t and belong to the set:

$$\Phi_+ \equiv \{ f : \mathbb{R}_+ \to \mathbb{R}_+ \; ; \; f \; C^\infty (\mathbb{R}_+) \} , \mathbb{R}_+ \equiv [0, \infty] \equiv$$

of the analytic function from R_+ to R_+ of infinite class. Thus, we have the formal relations of the generalized model:

$$Y(t) = \sum_{i=1}^{n} A_i(t) \qquad (2.1)$$

$$A_i(t) = b_i \, K_i(t) \qquad (i = 1, ..., n) \qquad (2.2)$$

$$\dot{K}_i(t) = \frac{d}{dt} \{ K_i(t) \} = \sum_{j=1}^{n} h_{ij} \, A_j(t) \qquad (i = 1, ..., n) \qquad (2.3)$$

$$\theta \leqslant h_{ij}, \gamma_j \leqslant 1 \; (i, j = 1, ..., n) \qquad (2.4)$$

$$b_i > \theta \qquad (i = 1, ..., n) \qquad (2.5)$$

The initial condition is:

$$A_i(\theta) = b_i \, K_i(\theta) \qquad (i = 1, ..., n) \qquad (2.6)$$

To solve the model in matrix form, let:

$$A(t) = \begin{bmatrix} A_1(t) \\ \cdot \\ \cdot \\ \cdot \\ A_n(t) \end{bmatrix} \tag{2.7}$$

$$K(t) = \begin{bmatrix} K_1(t) \\ \cdot \\ \cdot \\ \cdot \\ K_n(t) \end{bmatrix} \tag{2.8}$$

$$B = \begin{bmatrix} b_1 & & & \\ & \cdot & & o \\ o & & \cdot & \\ & & & \cdot \\ & & & b_n \end{bmatrix} \tag{2.9}$$

$$H = (h_{ij})\ nxn \tag{2.10}$$

be, respectively, the vector of sectoral production, the vector of sectoral capital stocks, the diagonal matrix of sectoral output/capital coefficients and the matrix of h_{ij} defined above.

The equations (2.2), (2.3) and (2.6) are now expressed as:

$$A(t) = BK(t) \tag{2.11}$$

$$\dot{K}(t) = HA(t) \tag{2.12}$$

$$A(\theta) = BK(\theta) \text{ or } K(\theta) = B^{-1} A(\theta) \tag{2.13}$$

Substituting (2.11) into (2.12), we have the following system of differential equations:

$$\dot{K}(t) = HB\ K(t)$$

The solution is given by the Taylor series[1]:

$$K(t) = [I_n + \sum_{m=1}^{\infty} (HB)^m \frac{t^m}{m!}]\ K(\theta) \tag{2.14}$$

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1 See Lancaster (1968) and Woods (1978).

where I_n is the nxn-identity matrix.

Substituting (2.14) into (2.11) and given the initial condition (2.13), we have:

$$A(t) = B[I_n + \sum_{m=1}^{\infty} (HB)^m \frac{t^m}{m!}] B^{-1} A(\theta) \qquad (2.15)$$

We denote by

$$|\cdot|_S : \Phi^n{}_+ \to \Phi_+$$

the norm of sum a functional vector in $\Phi^n{}_+$. Since $A_i(t) \in \Phi_+$ we have:

$$Y(t) = |A(t)|_S = \sum_{i=1}^{n} |A_i(t)| = \sum_{i=1}^{n} A_i(t)$$

We conclude that the total output of the economy $\in$ is given by the equation:[2]

$$Y(t) = |B[I_n + \sum_{m=1}^{\infty} (HB)^m \frac{t^m}{m!}] B^{-1} A(\theta)|_S \qquad (2.16)$$

3. BALANCED GROWTH PATH

Leontief's (1953) own dynamic model of growth produces a balanced path under certain conditions. The model we deal here is not that, but another closely related to DOSSO (1958) and, therefore, of the Leontief-type.

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2 In general: $Y(t) = |\Omega [I_n + \sum_{m=1}^{\infty} (H\Omega)^m \frac{t^m}{m!}] \Omega^{-1} A(0)|_S$

In particular, if n = 2 and B is the diagonal output/capital coefficient matrix, we have:

$$\Omega = \begin{cases} B \Rightarrow \text{Mahalanobis' model} \\ B^{-1} \Rightarrow \text{Feldman's model} \end{cases}$$

Such results are presented in Teixeira and Penaloza (1989).

Equation $\dot{K}(t) = HBK(t)$ may be written in its difference equation form. If we assume that stocks at the beginning of the period t must be sufficient to support the output level A (t) we have:

$$\Delta K(t) \leqq HBK(t) \tag{3.1}$$

To simplify notation let $(HB) = M$, provided that $| HB | \neq \theta$. Therefore:

$$M \Delta K \leqq K(t) \tag{3.2}$$

Suppose now that H is semi-positive indecomposable with a dominant root less than unit. In consequence of Hawkins-Simon (1949) condition $(I - H)^{-1}$ is an strictly positive matrix. It follows that M is semi-positive indecomposable.

Let us consider the conditions for balanced growth at the ratio $\mu = \dfrac{\Delta K_i(t)}{K_i(t)}$

for $i = 1, 2, ..., n$ and $t = 1, 2, ..., t$. In this case all sectors grow at the same ratio and at least one of then will fully utilize the respective capital capacity. Therefore:

$$M\Delta K(t) \langle \left(\frac{1}{\hat{\mu}}\right) \Delta K(t), \text{at least one equality}, K(t) \geqslant \theta \tag{3.3}$$

The only solution of (3.3) is $\dfrac{1}{\mu^*} = \lambda^*$, $K(t) = x^*$, where λ^* and x^* are, respectively, the dominant root and its associated characterictic vector[3]. Such solution gives $\Delta K(t)$ and $K(t)$ for all t since

$$K(t) = K(t-1) + \Delta K(t-1) \tag{3.4}$$

$$\therefore K(t) = (1 + \mu^*) K(t-1) = (1 + \mu^*)^t K(\theta) \tag{3.5}$$

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3 Thus, although we require equilibrium for only one commodity, we in fact obtain it for all, since the solution values give equalities throughout.

So we have the balanced growth path, provided that K (θ) is in the same proposition as X*. Since λ* is the Frobenius root the growth rate (which is the inverse of the root) is the smallest. No other growth path of this kind, however, have associated nonnegative stock vectors. On the other hand, if there are surpluses everywhere we have:

$$M\Delta K(t) \langle\ \langle (1/\hat{\mu})\,\Delta K(t) \tag{3.6}$$

In such situation, since $\Delta K(t) \geqslant \theta$, we can find some strictly positive matrix R such that $(M + R)\,\Delta K(t) \leqslant (1/\hat{\mu})\,\Delta K(t)$, at least for one equality. The solution is $1/\hat{\mu}_M = \mu^*(M + R)$ where $\lambda^*(M + R)$ denotes the Frobenius root associated with (M + R). It happens that both M and (M + R) are semi-positive indecomposable matrices with $(M + R) \rangle\ \rangle M$, so that $\lambda^*(M + R) \rangle \lambda^*_M$ and $\hat{\mu} \langle \mu$. Therefore the rate of balanced growth with surplus stocks is less the balanced equilibrium growth rate μ^*.

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SINOPSE

UM MODELO LINEAR E SEU CAMINHO DE CRESCIMENTO EQUILIBRADO

Neste artigo formaliza-se um modelo linear de crescimento onde a distinção entre bens de investimento e bens de consumo não é estabeleci-

da, a priori, mas baseia-se no tipo de demanda dos agentes econômicos. Usa-se uma estrutura de insumo-produto com n setores e mostra-se a existência de crescimento balanceado no contexto de um modelo dinâmico fechado tipo Leontief, semelhante ao abordado por DOSSO.